空なる証明の道

Ku naru shoh-mei no michi

The way of the empty proof

L’art de la preuve vide

If you have the right definition you have a simpler proof.

If you have the right data structures, you have a simpler algorithm.

If you have the right…..

Power of Elimination

除去の力

Tarski's Meta-theorem

To any formula $\Phi(X_1, X_2, \ldots\ldots, X_m)$ in the vocabulary {0,1,+,.,=,<} one can effectively associate two objects:

(i) a quantifier free formula $\theta(X_1, \ldots\ldots, X_m)$ in the same vocabulary and

(ii) a proof of the equivalence $\Phi \leftarrow\rightarrow \theta$ that uses the axioms for real closed fields.

$\exists x \quad ax^2 + b + c = 0$

If and only if

$b^2 - 4ac > 0$

Resolution: Tarski's Meta-theorem for Logic

Elimination not restricted to algebra and geometry

$3(2x - y \leq 5)$

$6x - 3y \leq 15$

$2(-3x + y \leq -1)$

$-6x + 2y \leq -2$

$0x \quad -7y \leq 13$

P(a) V Q(x)

P(x) V Q(x)

¬P(a) V R(y)

x=a

Q(a) V R(y)

Q(x) V R(y)

Used in automated theorem proving (algebra, geometry & logic)

But…

However…

Can be used in special cases to prove theorems by hand

$F_0 \Rightarrow F_1 \Rightarrow F_2 \Rightarrow F_3 \Rightarrow F_4 \ldots\ldots \Rightarrow F_n$

P P P P P P?

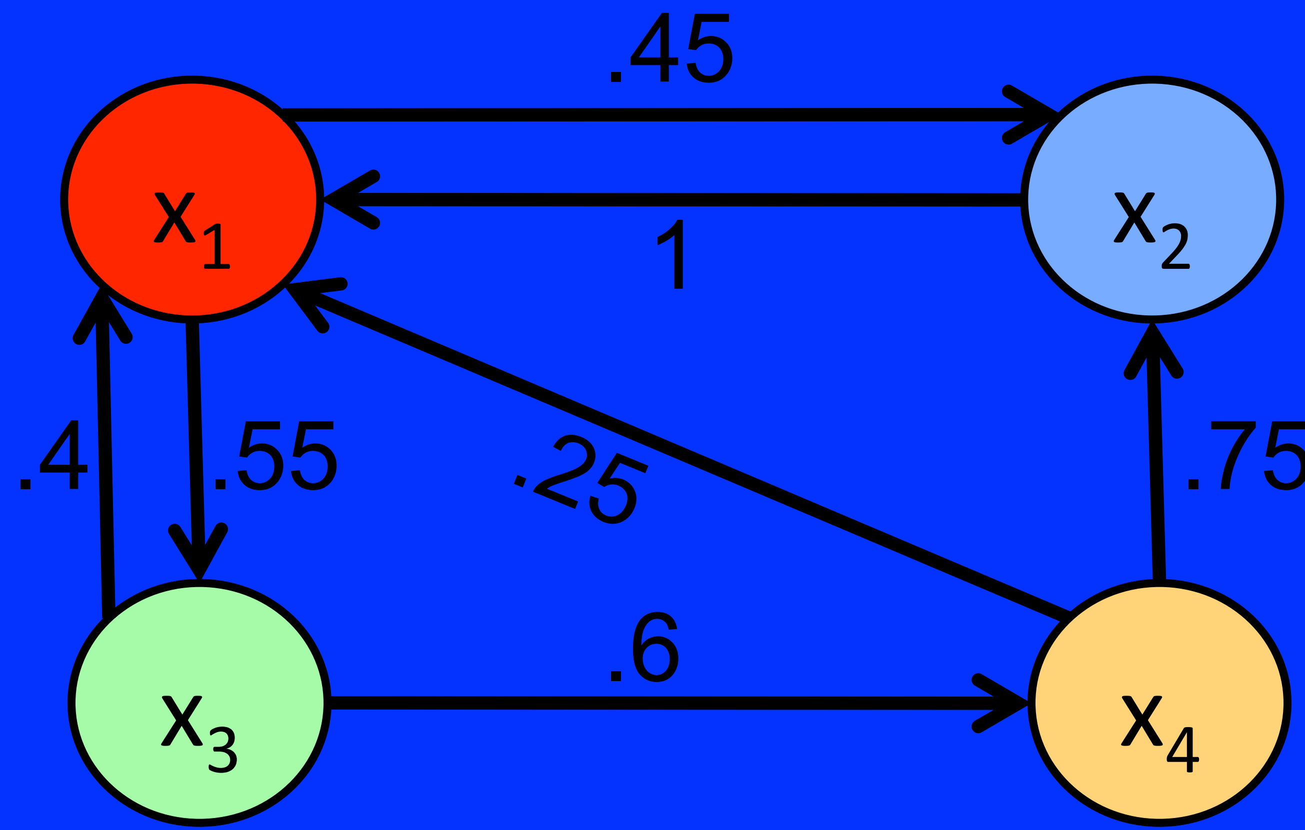
The elimination of the proof is an ideal seldom reached

Elimination gives us the heart of the proof

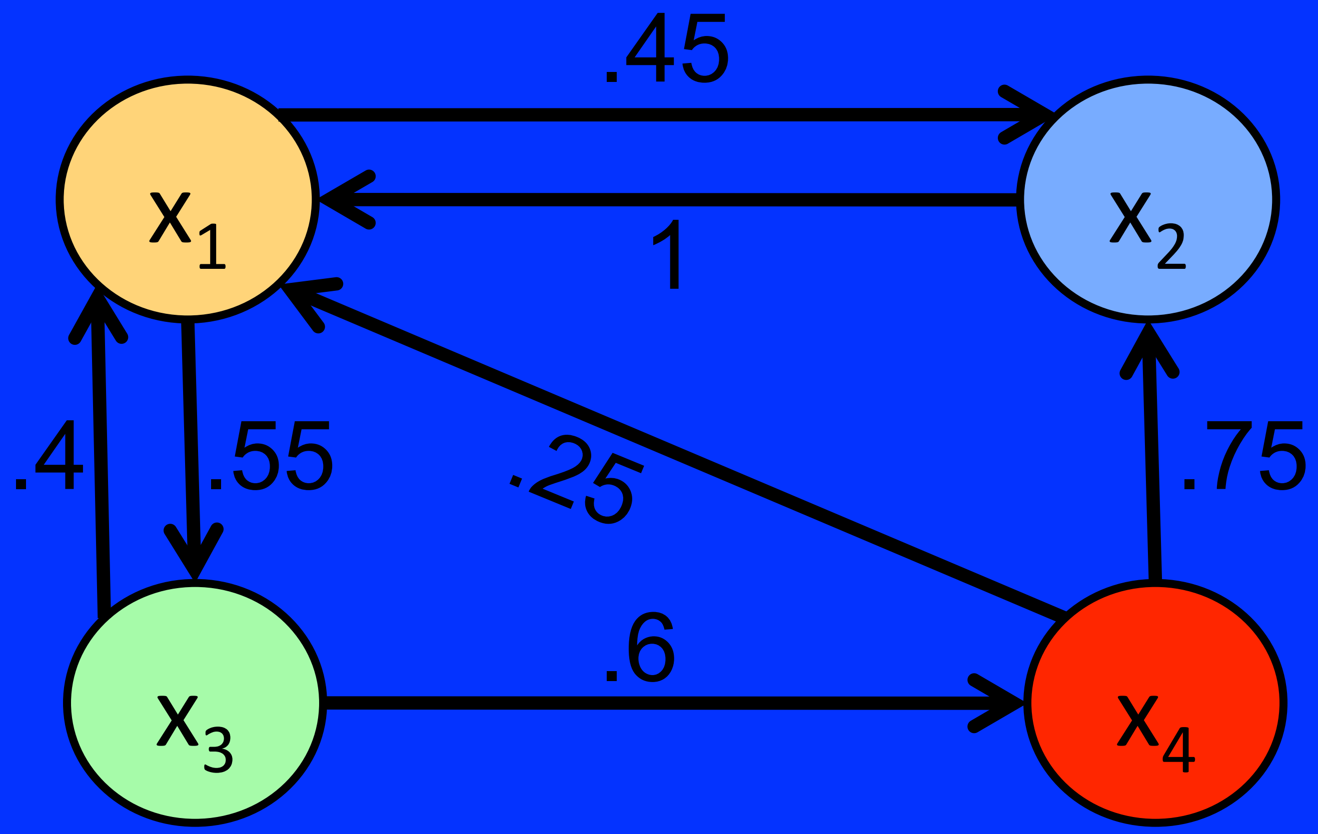
Markov's Ergodic Theorem (1906)

Any irreducible, finite, aperiodic Markov Chain has all states Ergodic (reachable at any time in the future) and has a unique stationary distribution, which is a probability vector.

Probabilities after 15 steps	30 steps	100 steps	500 steps	1000 steps
X1 = .33	X1 = .33	X1 = .37	X1 = .38	X1 = .38
X2 = .26	X2 = .26	X2 = .29	X2 = .28	X2 = .28
X3 = .26	X3 = .23	X3 = .21	X3 = .21	X3 = .21
X4 = .13	X4 = .16	X4 = .13	X4 = .13	X4 = .13



Probabilities after 15 steps	30 steps	100 steps	500 steps	1000 steps
X1 = .46	X1 = .36	X1 = .38	X1 = .38	X1 = .38
X2 = .20	X2 = .26	X2 = .28	X2 = .28	X2 = .28
X3 = .26	X3 = .23	X3 = .21	X3 = .21	X3 = .21
X4 = .06	X4 = .13	X4 = .13	X4 = .13	X4 = .13



$$p_{11}x_1 + p_{21}x_2 + p_{31}x_3 = x_1$$
$$p_{12}x_1 + p_{22}x_2 + p_{32}x_3 = x_2$$
$$p_{13}x_1 + p_{23}x_2 + p_{33}x_3 = x_3$$
$$\sum x_i = 1$$
$$x_i \geq 0$$

You can view the problem in three different ways:

- Principal eigenvector problem
- Classical Linear Programming problem
- Elimination problem

Difficulties as an Eigenvector Problem

- Notion of convergence
- Deal with complex numbers
- Uniqueness of solution

- Need to use theorems: Perron-Frobenius, Chapman, Kolmogoroff, Cauchy… and/or restrictive hypotheses….

Symbolic Gaussian Elimination

System of two variables:

$$p_{11}x_1 + p_{21}x_2 = x_1$$

$$p_{12}x_1 + p_{22}x_2 = x_2$$

$$\sum x_i = 1$$

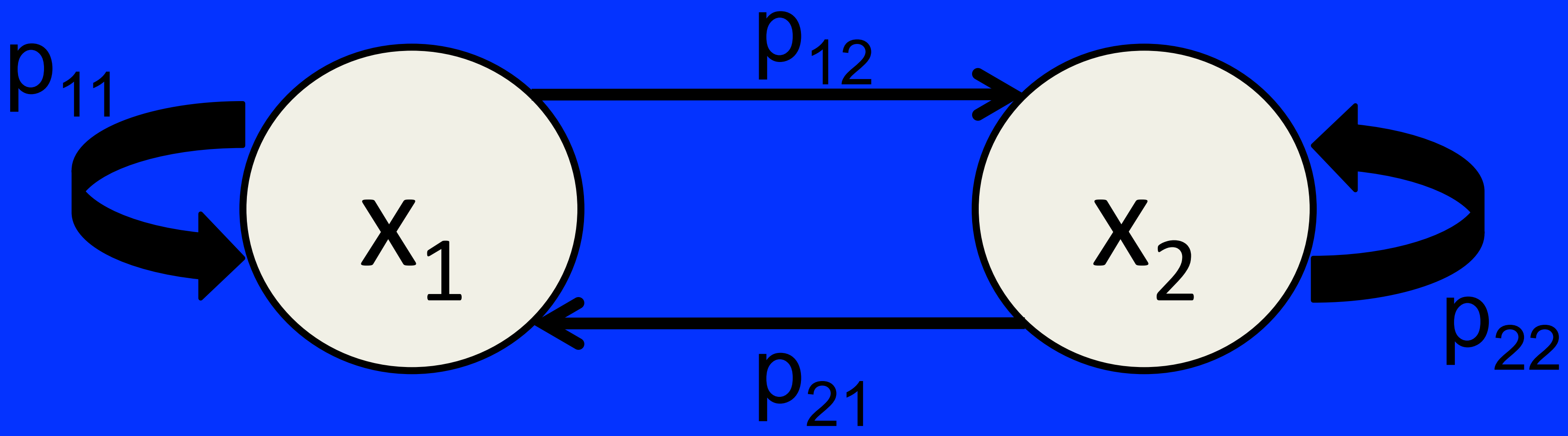
With Maple we find:

$$x_1 = p_{21}/(p_{21} + p_{12})$$

$$x_2 = p_{12}/(p_{21} + p_{12})$$

$$x_1 = p_{21}/(p_{21} + p_{12})$$

$$x_2 = p_{12}/(p_{21} + p_{12})$$



Symbolic Gaussian Elimination

Three variables:

$$p_{11}x_1 + p_{21}x_2 + p_{31}x_3 = x_1$$

$$p_{12}x_1 + p_{22}x_2 + p_{32}x_3 = x_2$$

$$p_{13}x_1 + p_{23}x_2 + p_{33}x_3 = x_3$$

$$\sum x_i = 1$$

With Maple we find:

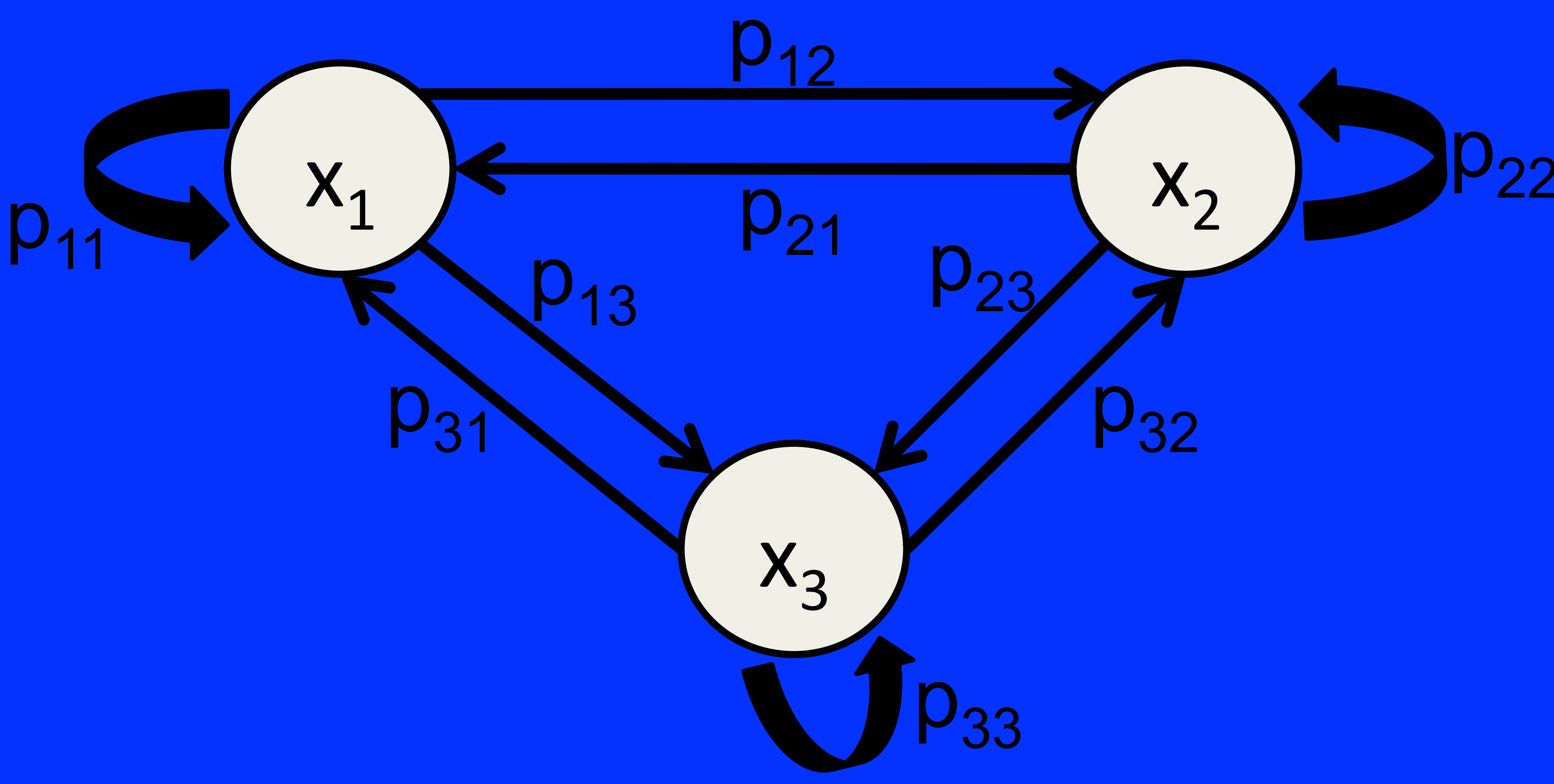
$$x_1 = (p_{31}p_{21} + p_{31}p_{23} + p_{32}p_{21}) / \Sigma$$

$$x_2 = (p_{13}p_{32} + p_{12}p_{31} + p_{12}p_{32}) / \Sigma$$

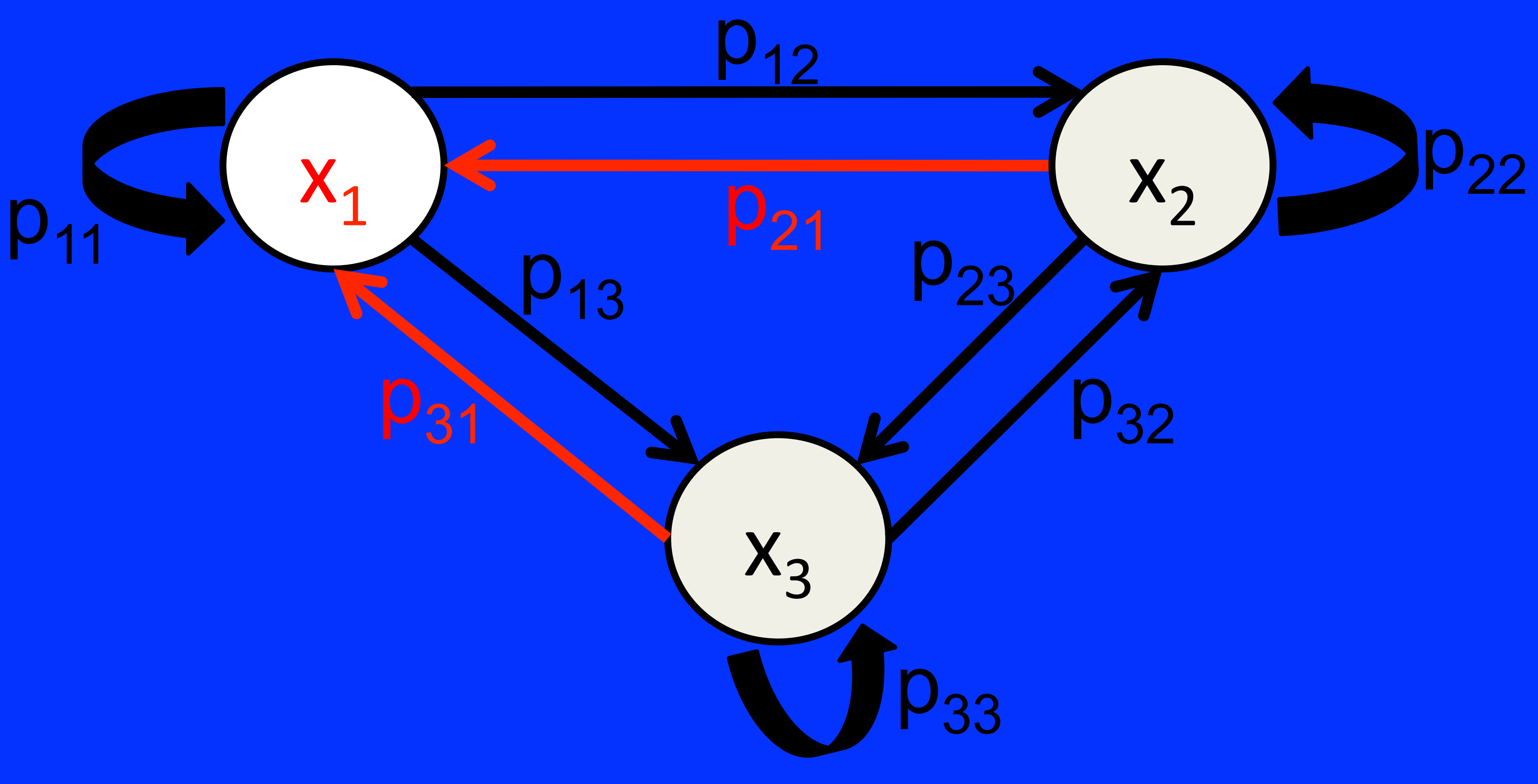
$$x_3 = (p_{13}p_{21} + p_{12}p_{23} + p_{13}p_{23}) / \Sigma$$

$$\Sigma = (p_{31}p_{21} + p_{31}p_{23} + p_{32}p_{21} + p_{13}p_{32} + p_{12}p_{31} + p_{12}p_{32} + p_{13}p_{21} + p_{12}p_{23} + p_{13}p_{23})$$

$$x_1 = (p_{31}p_{21} + p_{23}p_{31} + p_{32}p_{21}) / \Sigma$$
$$x_2 = (p_{13}p_{32} + p_{31}p_{12} + p_{12}p_{32}) / \Sigma$$
$$x_3 = (p_{21}p_{13} + p_{12}p_{23} + p_{13}p_{23}) / \Sigma$$



$$x_1 = (p_{31}p_{21} + p_{23}p_{31} + p_{32}p_{21}) / \Sigma$$
$$x_2 = (p_{13}p_{32} + p_{31}p_{12} + p_{12}p_{32}) / \Sigma$$
$$x_3 = (p_{21}p_{13} + p_{12}p_{23} + p_{13}p_{23}) / \Sigma$$



$$x_1 = (p_{31}p_{21} + p_{23}p_{31} + p_{32}p_{21}) / \Sigma$$
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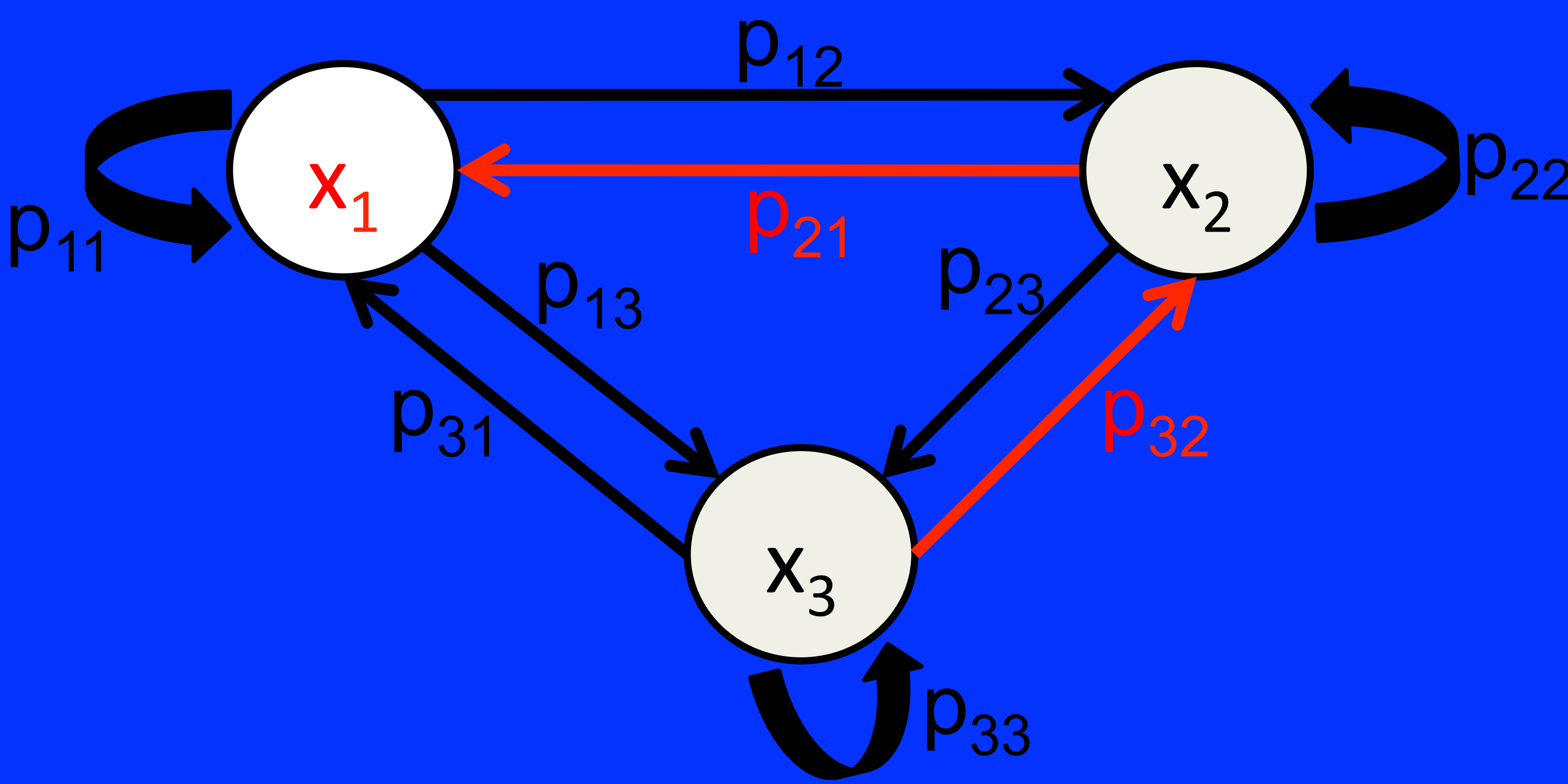

$$x_1 = (p_{31}p_{21} + p_{23}p_{31} + p_{32}p_{21}) / \Sigma$$

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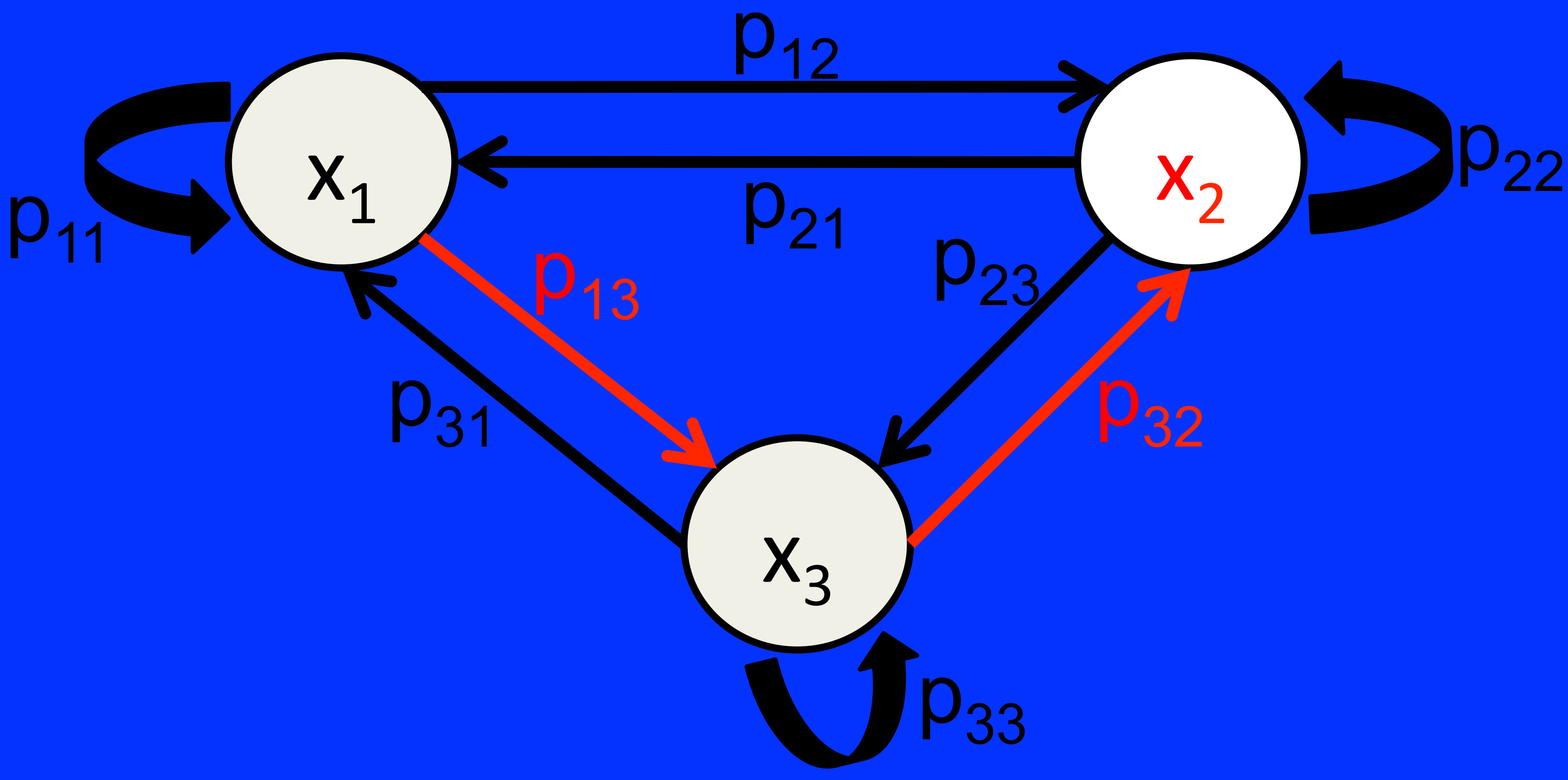

$$x_1 = (p_{31}p_{21} + p_{23}p_{31} + p_{32}p_{21}) / \Sigma$$
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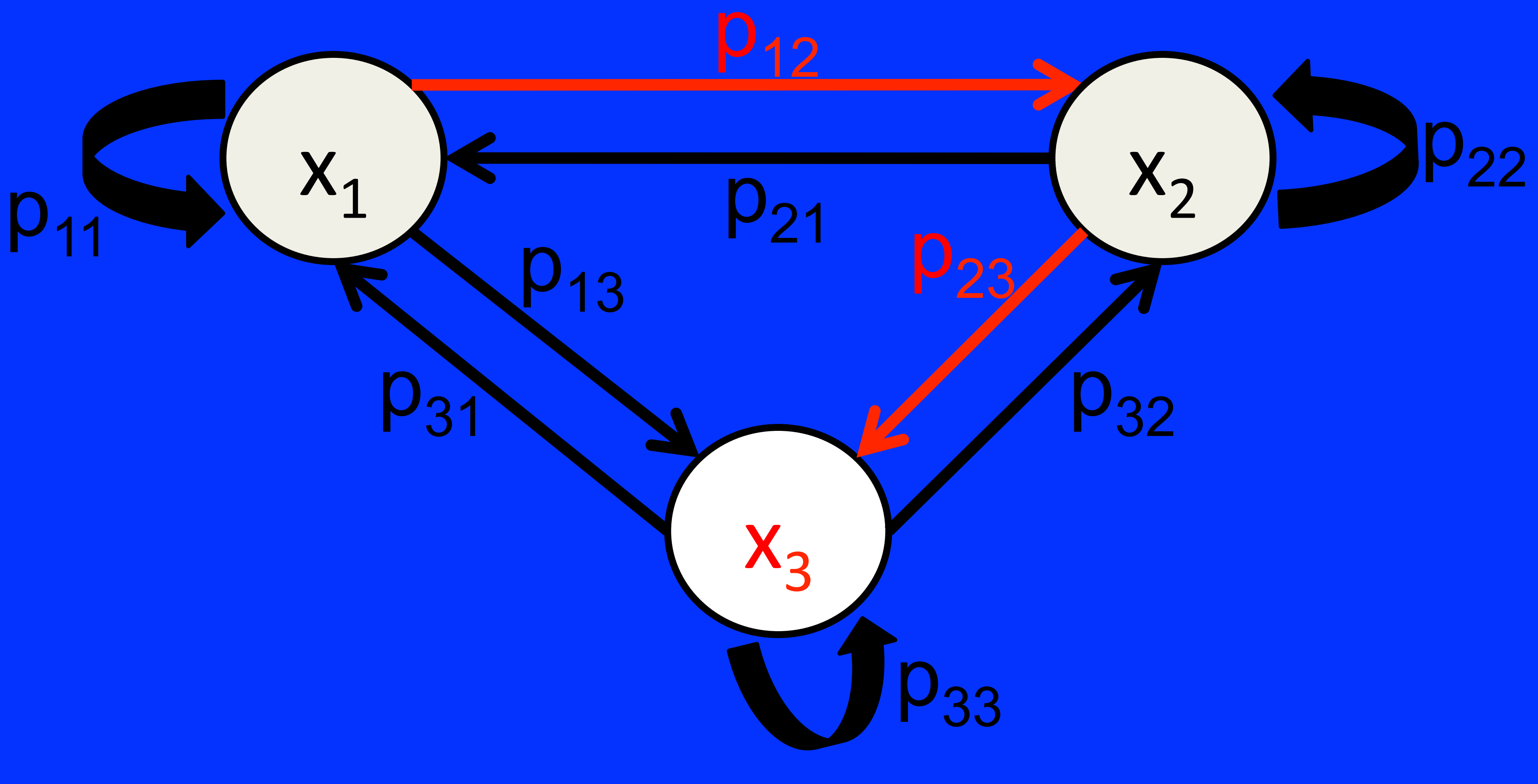


Do you see the LIGHT?

James Brown (The Blues Brothers)

Symbolic Gaussian Elimination

System of four variables:

$$p_{21}x_2 + p_{31}x_3 + p_{41}x_4 = x_1$$

$$p_{12}x_1 + p_{42}x_4 = x_2$$

$$p_{13}x_1 = x_3$$

$$p_{34}x_3 = x_4$$

$$\sum x_i = 1$$

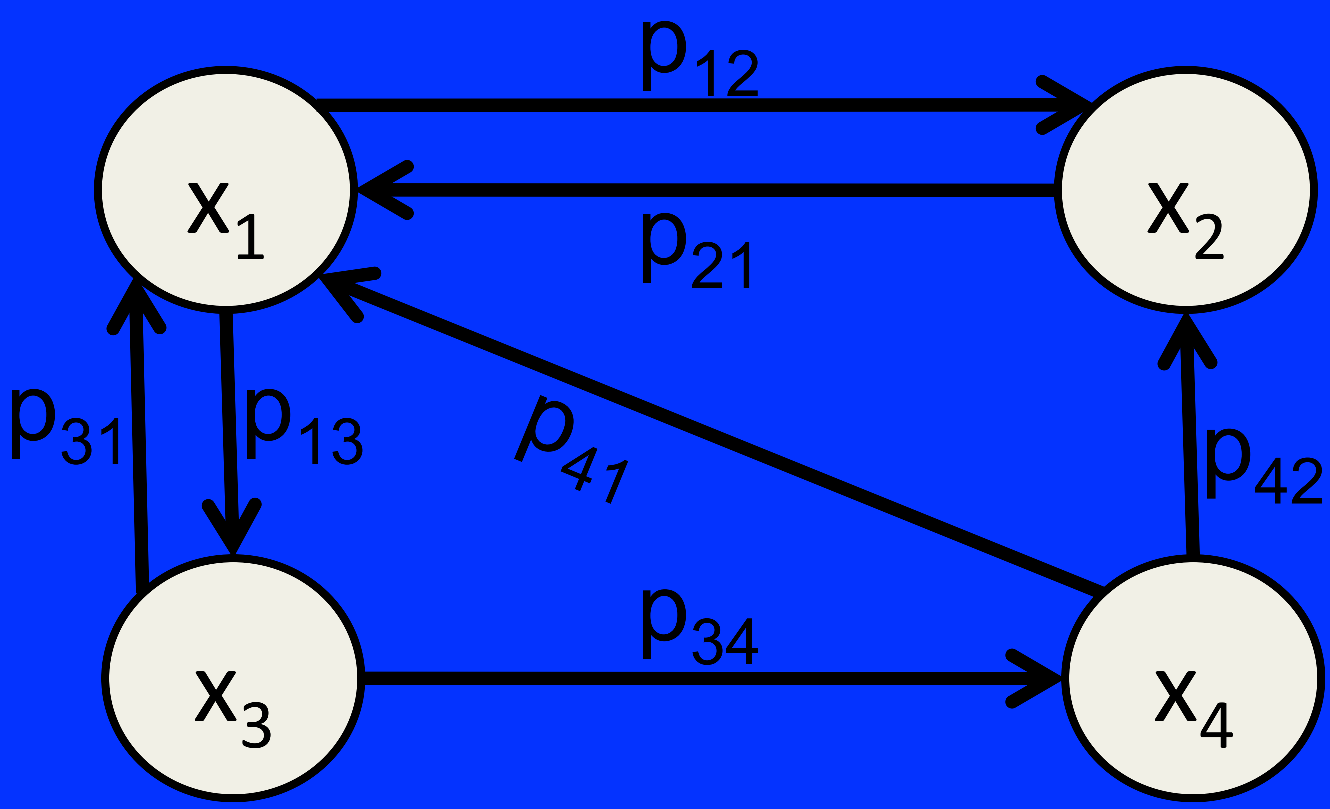
With Maple we find:

$x_1 = p_{21}p_{34}p_{41} + p_{34}p_{42}p_{21} + p_{21}p_{31}p_{41} + p_{31}p_{42}p_{21} / \Sigma$

$x_2 = p_{31}p_{41}p_{12} + p_{31}p_{42}p_{12} + p_{34}p_{41}p_{12} + p_{34}p_{42}p_{12} + p_{13}p_{34}p_{42} / \Sigma$

$x_3 = p_{41}p_{21}p_{13} + p_{42}p_{21}p_{13} / \Sigma$

$x_4 = p_{21}p_{13}p_{34} / \Sigma$



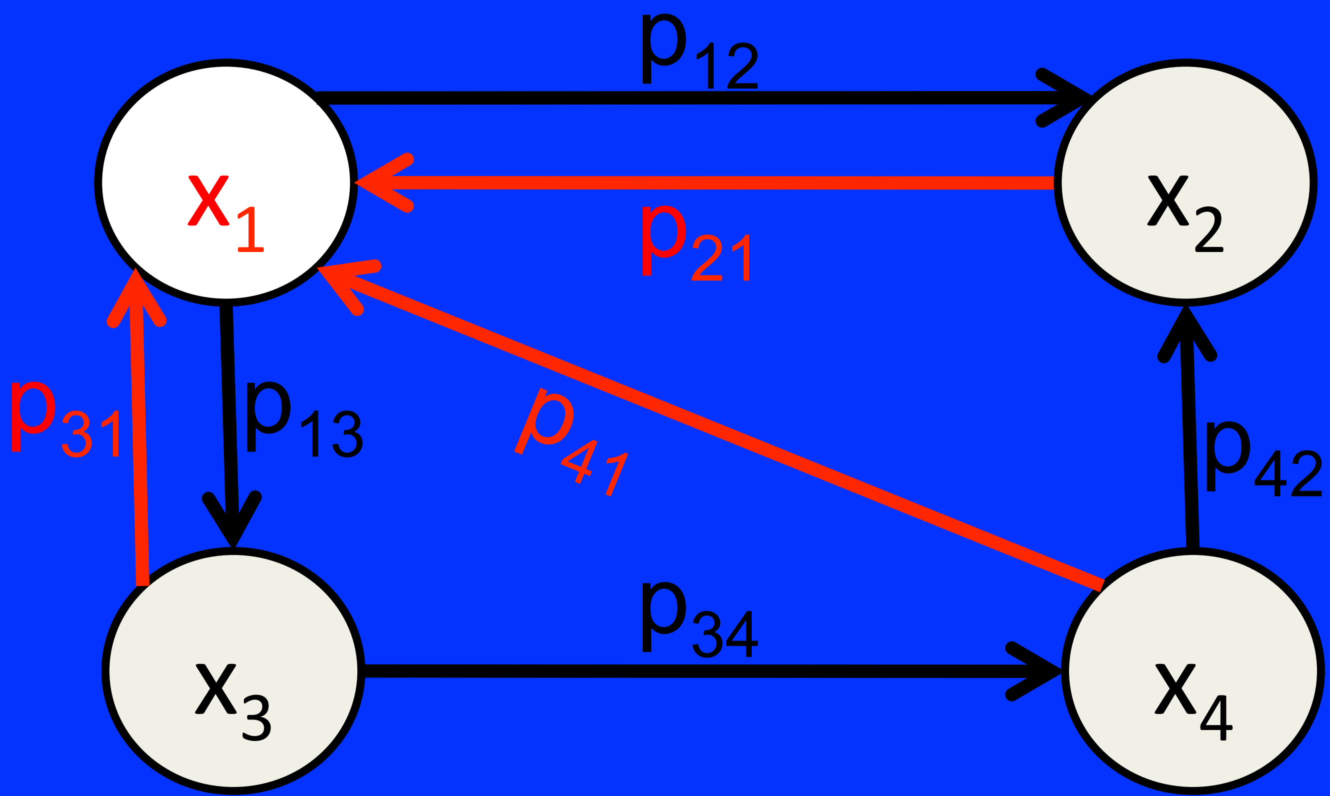
$\Sigma = p_{21}p_{34}p_{41} + p_{34}p_{42}p_{21} + p_{21}p_{31}p_{41} + p_{31}p_{42}p_{21} + p_{31}p_{41}p_{12} + p_{31}p_{42}p_{12} + p_{34}p_{41}p_{12} + p_{34}p_{42}p_{12} + p_{13}p_{34}p_{42} + p_{41}p_{21}p_{13} + p_{42}p_{21}p_{13} + p_{21}p_{13}p_{34}$

$x_1 = p_{21}p_{34}p_{41} + p_{34}p_{42}p_{21} + p_{21}p_{31}p_{41} + p_{31}p_{42}p_{21} / \Sigma$

$x_2 = p_{31}p_{41}p_{12} + p_{31}p_{42}p_{12} + p_{34}p_{41}p_{12} + p_{34}p_{42}p_{12} + p_{13}p_{34}p_{42} / \Sigma$

$x_3 = p_{41}p_{21}p_{13} + p_{42}p_{21}p_{13} / \Sigma$

$x_4 = p_{21}p_{13}p_{34} / \Sigma$



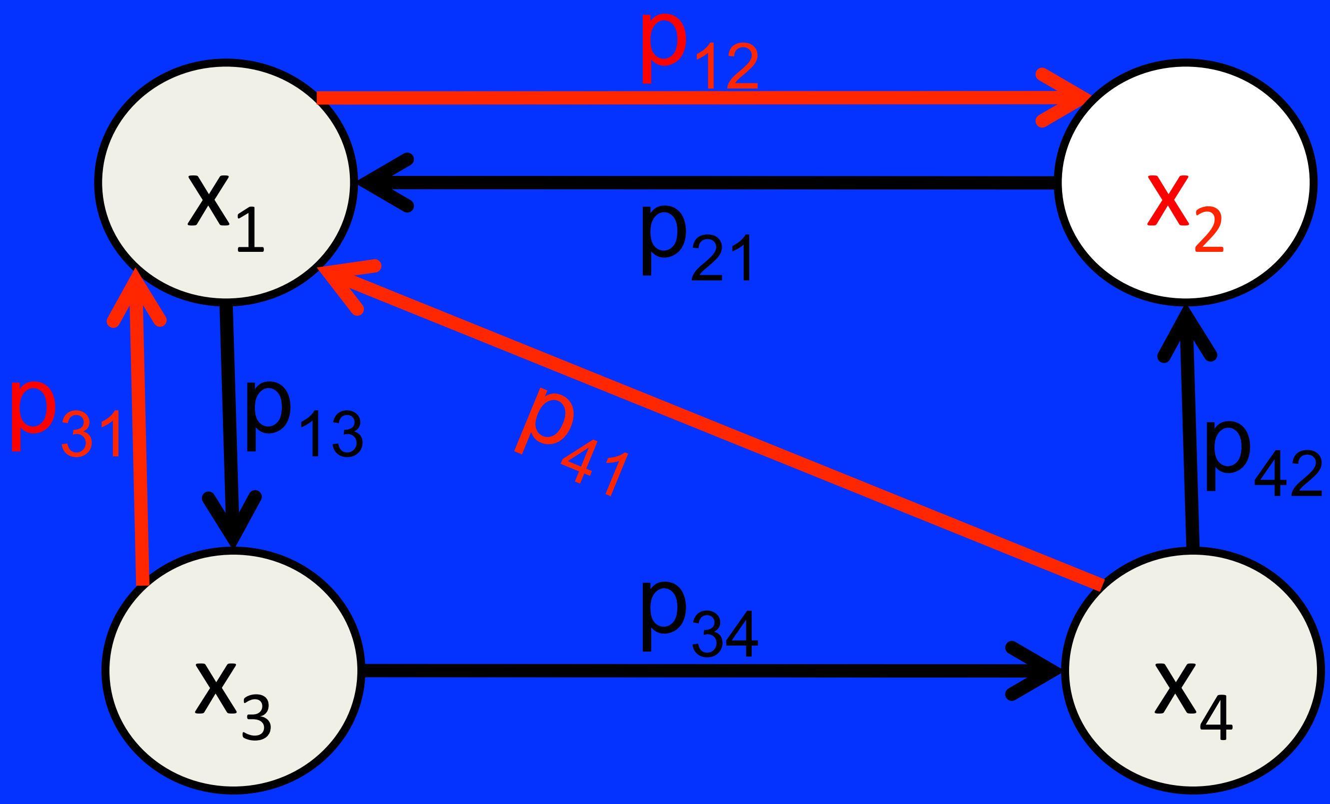
$\Sigma = p_{21}p_{34}p_{41} + p_{34}p_{42}p_{21} + p_{21}p_{31}p_{41} + p_{31}p_{42}p_{21} + p_{31}p_{41}p_{12} + p_{31}p_{42}p_{12} + p_{34}p_{41}p_{12} + p_{34}p_{42}p_{12} + p_{13}p_{34}p_{42} + p_{41}p_{21}p_{13} + p_{42}p_{21}p_{13} + p_{21}p_{13}p_{34}$

$x_1 = p_{21}p_{34}p_{41} + p_{34}p_{42}p_{21} + p_{21}p_{31}p_{41} + p_{31}p_{42}p_{21} / \Sigma$

$x_2 = p_{31}p_{41}p_{12} + p_{31}p_{42}p_{12} + p_{34}p_{41}p_{12} + p_{34}p_{42}p_{12} + p_{13}p_{34}p_{42} / \Sigma$

$x_3 = p_{41}p_{21}p_{13} + p_{42}p_{21}p_{13} / \Sigma$

$x_4 = p_{21}p_{13}p_{34} / \Sigma$



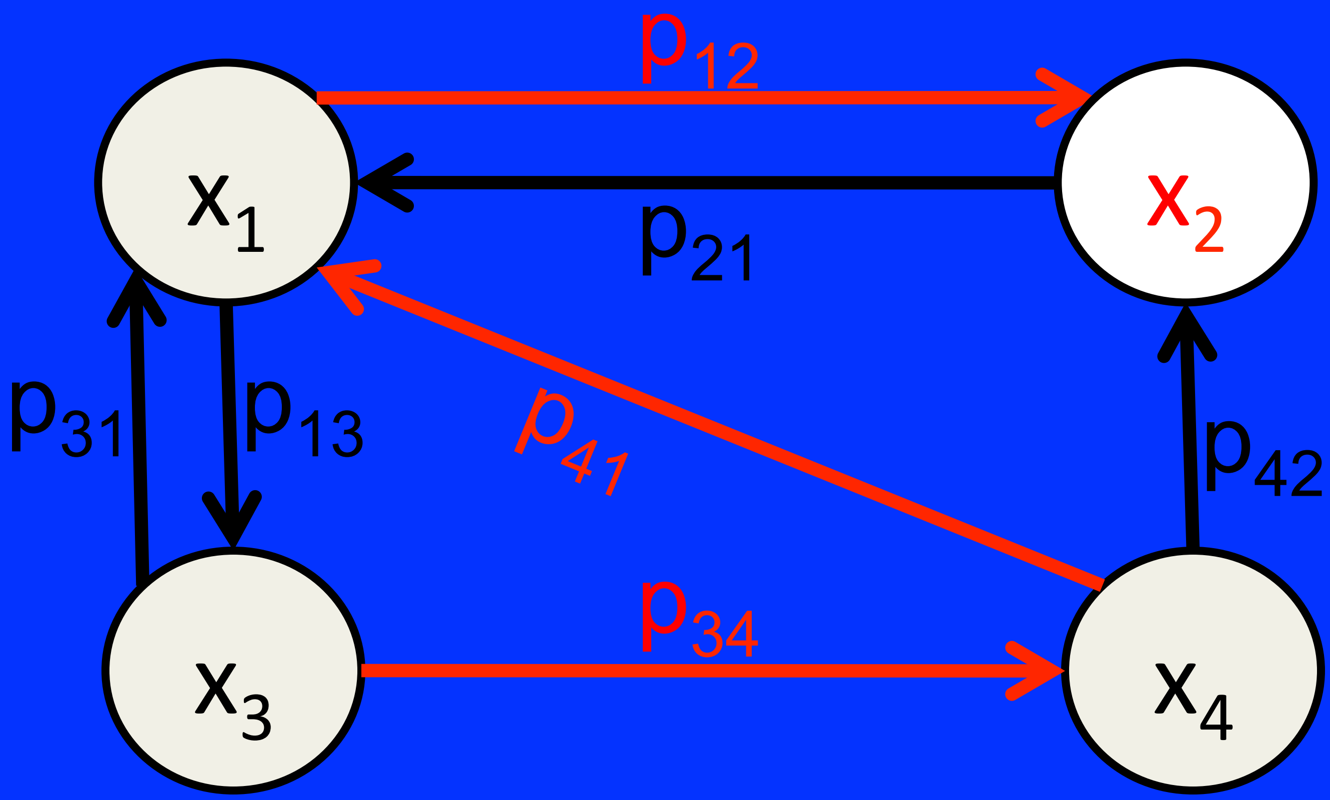
$\Sigma = p_{21}p_{34}p_{41} + p_{34}p_{42}p_{21} + p_{21}p_{31}p_{41} + p_{31}p_{42}p_{21} + p_{31}p_{41}p_{12} + p_{31}p_{42}p_{12} + p_{34}p_{41}p_{12} + p_{34}p_{42}p_{12} + p_{13}p_{34}p_{42} + p_{41}p_{21}p_{13} + p_{42}p_{21}p_{13} + p_{21}p_{13}p_{34}$

$x_1 = p_{21}p_{34}p_{41} + p_{34}p_{42}p_{21} + p_{21}p_{31}p_{41} + p_{31}p_{42}p_{21} / \Sigma$

$x_2 = p_{31}p_{41}p_{12} + p_{31}p_{42}p_{12} + p_{34}p_{41}p_{12} + p_{34}p_{42}p_{12} + p_{13}p_{34}p_{42} / \Sigma$

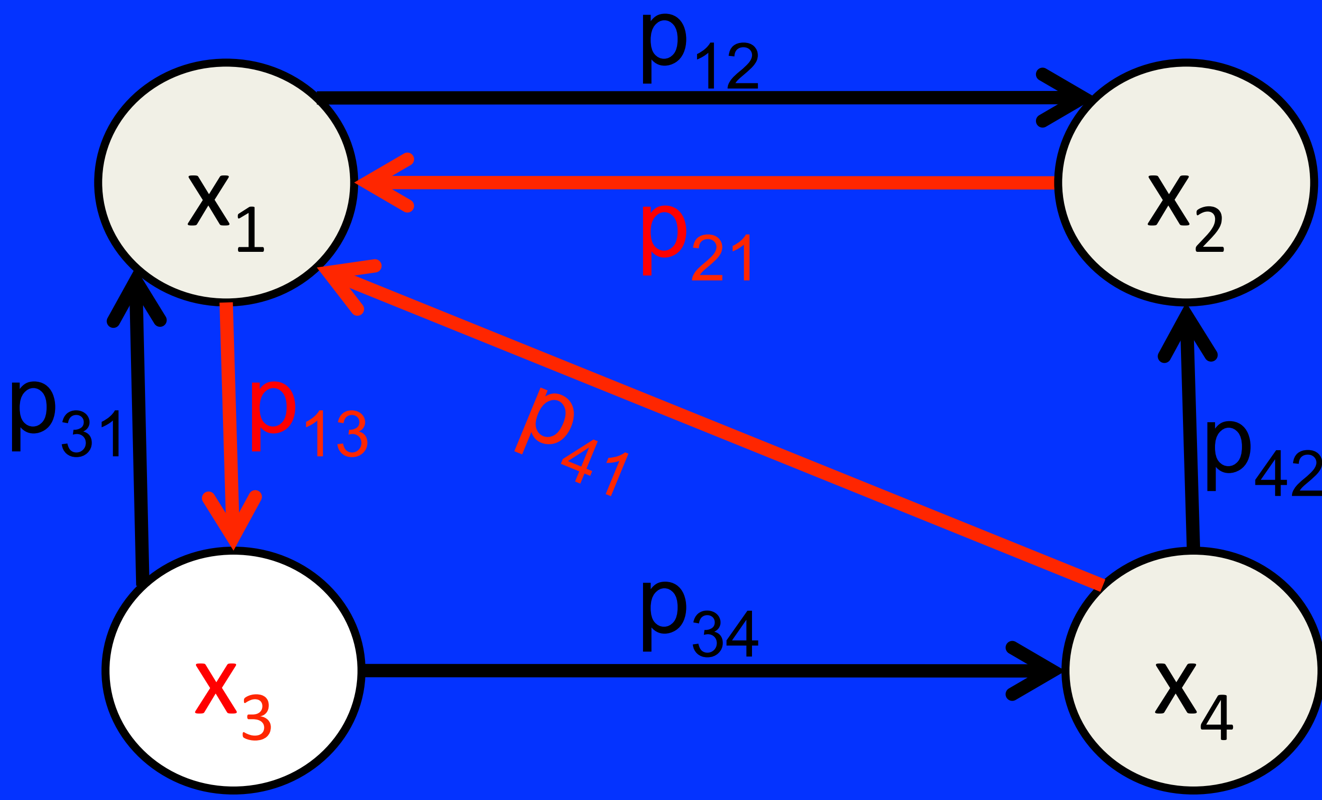
$x_3 = p_{41}p_{21}p_{13} + p_{42}p_{21}p_{13} / \Sigma$

$x_4 = p_{21}p_{13}p_{34} / \Sigma$



$\Sigma = p_{21}p_{34}p_{41} + p_{34}p_{42}p_{21} + p_{21}p_{31}p_{41} + p_{31}p_{42}p_{21} + p_{31}p_{41}p_{12} + p_{31}p_{42}p_{12} + p_{34}p_{41}p_{12} + p_{34}p_{42}p_{12} + p_{13}p_{34}p_{42} + p_{41}p_{21}p_{13} + p_{42}p_{21}p_{13} + p_{21}p_{13}p_{34}$

$x_1 = p_{21}p_{34}p_{41} + p_{34}p_{42}p_{21} + p_{21}p_{31}p_{41} + p_{31}p_{42}p_{21} / \Sigma$
$x_2 = p_{31}p_{41}p_{12} + p_{31}p_{42}p_{12} + p_{34}p_{41}p_{12} + p_{34}p_{42}p_{12} + p_{13}p_{34}p_{42} / \Sigma$
$x_3 = p_{41}p_{21}p_{13} + p_{42}p_{21}p_{13} / \Sigma$
$x_4 = p_{21}p_{13}p_{34} / \Sigma$



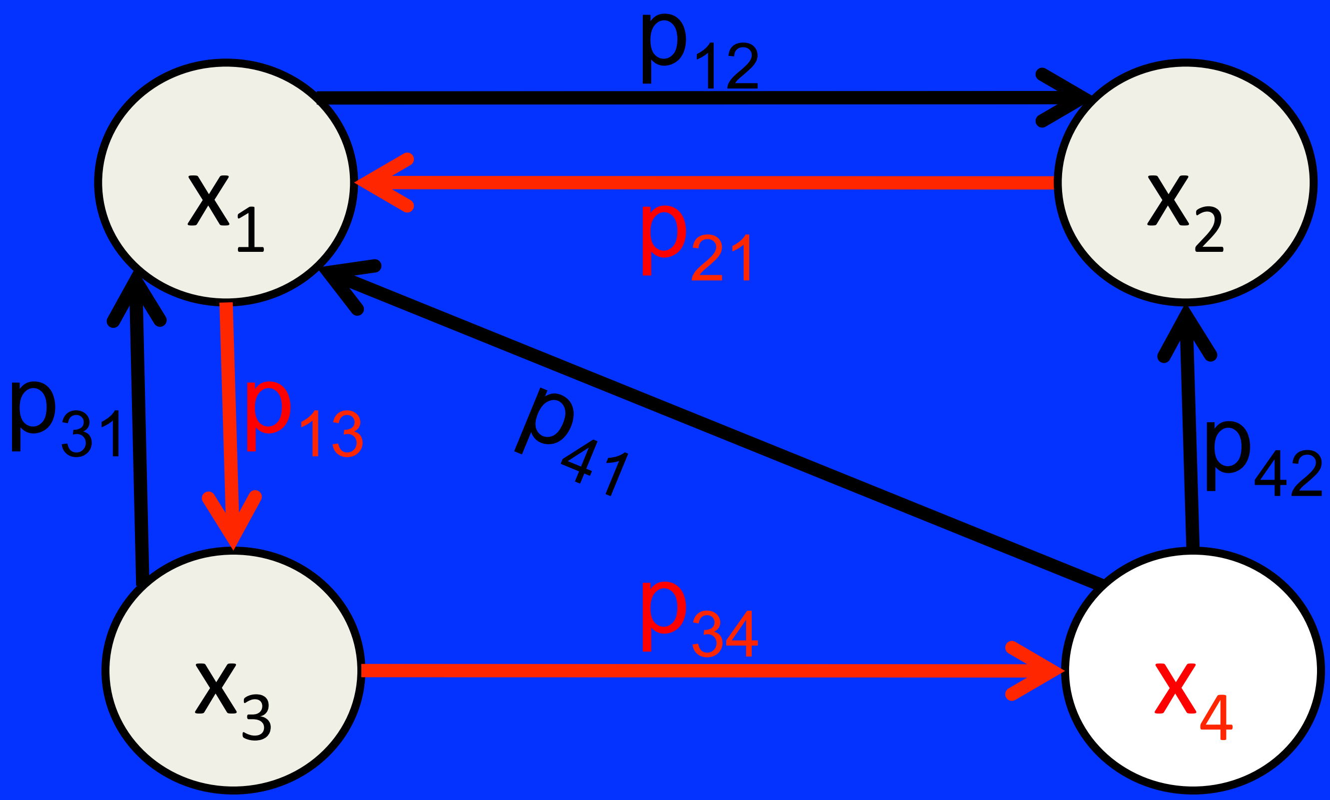
$\Sigma = p_{21}p_{34}p_{41} + p_{34}p_{42}p_{21} + p_{21}p_{31}p_{41} + p_{31}p_{42}p_{21} + p_{31}p_{41}p_{12} + p_{31}p_{42}p_{12} + p_{34}p_{41}p_{12} + p_{34}p_{42}p_{12} + p_{13}p_{34}p_{42} + p_{41}p_{21}p_{13} + p_{42}p_{21}p_{13} + p_{21}p_{13}p_{34}$

$x_1 = p_{21}p_{34}p_{41} + p_{34}p_{42}p_{21} + p_{21}p_{31}p_{41} + p_{31}p_{42}p_{21} / \Sigma$

$x_2 = p_{31}p_{41}p_{12} + p_{31}p_{42}p_{12} + p_{34}p_{41}p_{12} + p_{34}p_{42}p_{12} + p_{13}p_{34}p_{42} / \Sigma$

$x_3 = p_{41}p_{21}p_{13} + p_{42}p_{21}p_{13} / \Sigma$

$x_4 = p_{21}p_{13}p_{34} / \Sigma$



$\Sigma = p_{21}p_{34}p_{41} + p_{34}p_{42}p_{21} + p_{21}p_{31}p_{41} + p_{31}p_{42}p_{21} + p_{31}p_{41}p_{12} + p_{31}p_{42}p_{12} + p_{34}p_{41}p_{12} + p_{34}p_{42}p_{12} + p_{13}p_{34}p_{42} + p_{41}p_{21}p_{13} + p_{42}p_{21}p_{13} + p_{21}p_{13}p_{34}$

Ergodic Theorem Revisited

If there exists a reverse spanning tree in a graph of the Markov chain associated to a stochastic system, then:

(a)the stochastic system admits the following probability vector as a solution:

$$\left\{x_i = \frac{W(i)}{\sum_j W(j)}\right\} i = 1, n$$

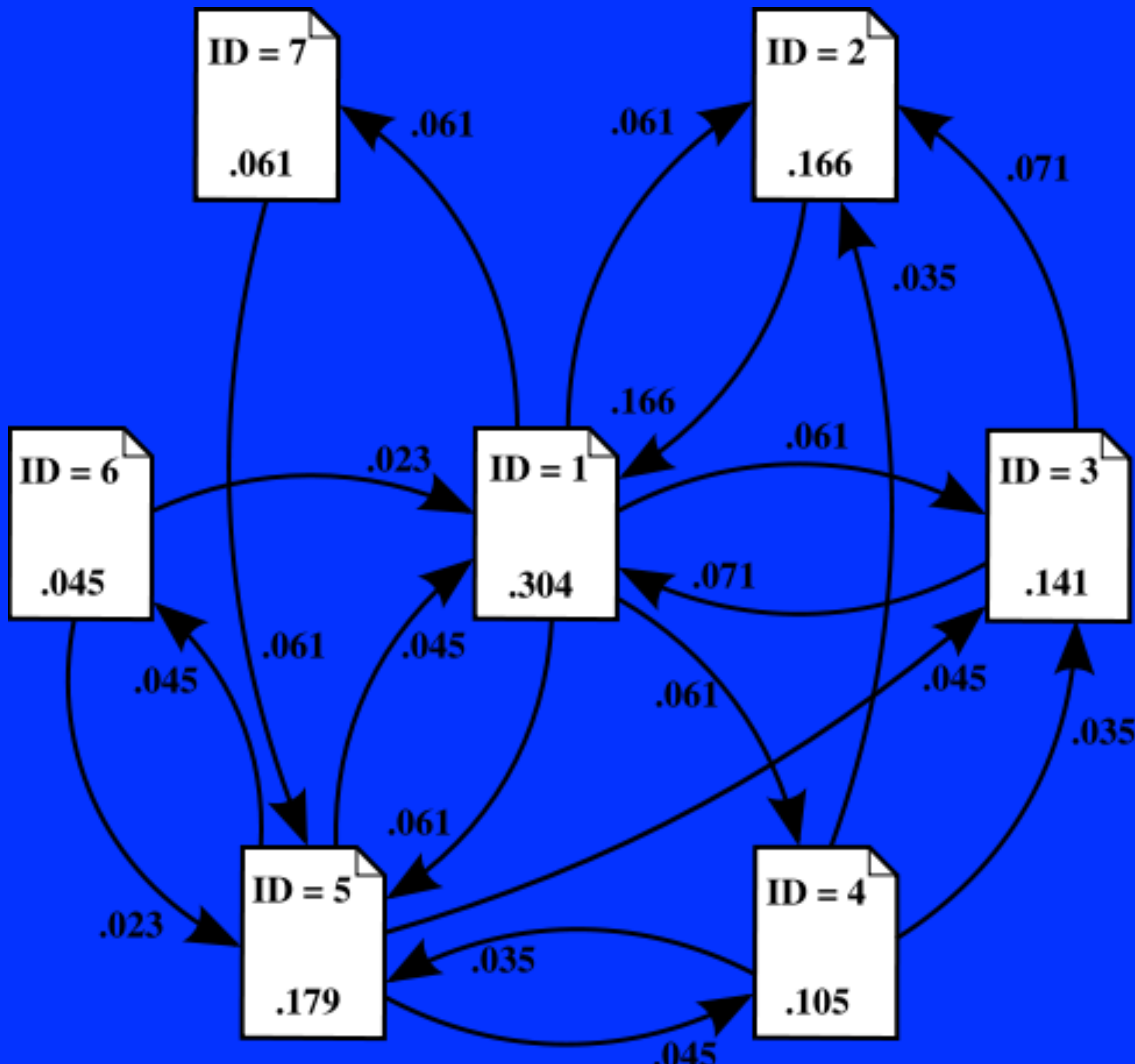
(b) the solution is unique.

(c) the conditions $\{x_i \geq 0\}_{i=1,n}$ are redundant and the solution can be computed by Gaussian elimination.

Internet Sites

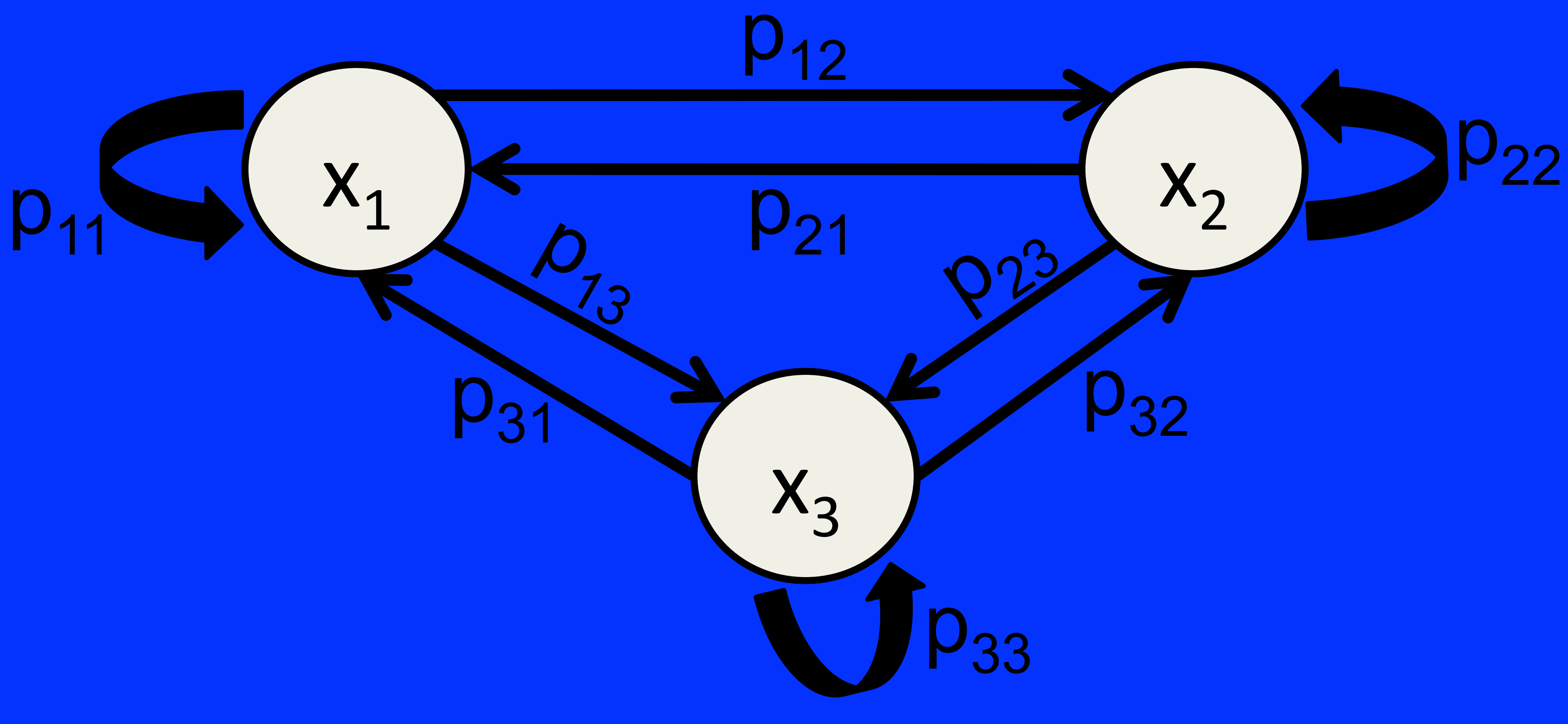
- Kleinberg
- Google
- SALSA

- In degree heuristic



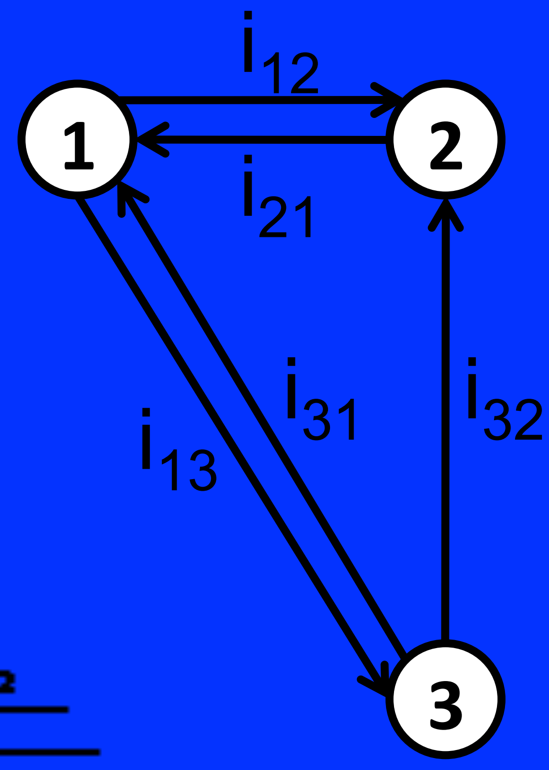
Markov Chain as a Conservation System

$$p_{11}x_1 + p_{21}x_2 + p_{31}x_3 = x_1(p_{11} + p_{12} + p_{13})$$
$$p_{12}x_1 + p_{22}x_2 + p_{32}x_3 = x_2(p_{21} + p_{22} + p_{23})$$
$$p_{13}x_1 + p_{23}x_2 + p_{33}x_3 = x_3(p_{31} + p_{32} + p_{33})$$



Kirchoff’s Current Law

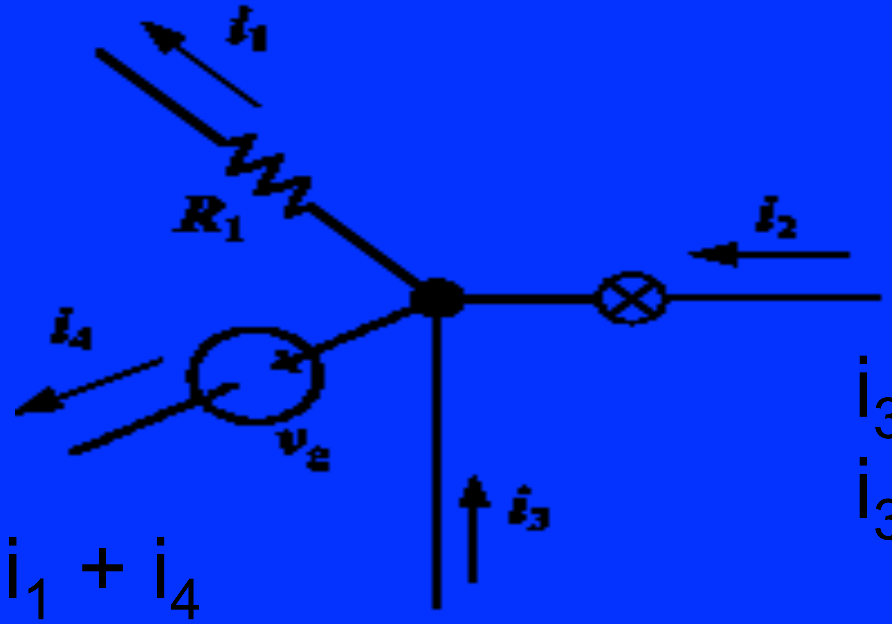
- The sum of currents flowing towards a node is equal to the sum of currents flowing away from the node.



$i_{31} + i_{21} = i_{12} + i_{13}$

$i_{32} + i_{12} = i_{21}$

$i_{13} = i_{31} + i_{32}$



$i_3 + i_2 = i_1 + i_4$

Kirchoff's Matrix Tree Theorem (1847)

For an n-vertex digraph, define an n x n matrix A such that A[i,j] = 1 if there is an edge from i to j, for all $i \neq j$, and the diagonal entries are such that the row sums are 0.

Let A(k) be the matrix obtained from A by deleting row k and column k. Then the absolute value of the determinant of A(k) is the number of spanning trees rooted at k (edges directed towards vertex k)

Two theorems for the price of one!!

Differences

- Kirchoff theorem perform n gaussian eliminations
- Revised version – only two gaussian

Minimax Theorem

- Fundamental Theorem in Game Theory
- Von Neumann & Kuhn

- Minimax Theorem brings certainty into the world of probabilistic game theory.

- Applications in Computer Science, Economics & Business, Biology, etc.

Minimax Theorem

$\max(x + 2y)$

$z = x + 2y$

$-3x + 7y \le 2$ / $2z \le 2/2$

$2x + 3y \le 4$ / $3z \le 8/3$

$4x - 3y \le 1$ / $0 \le z$

$10z \le 19/10$

$11z \le 30/11$

$-2/3 \le z$

$$0 \le z \le 1$$

max value z can take is min value of the right hand side

min value z can take is max value of the left hand side

Duality Theorem

If the primal problem has an optimal solution,

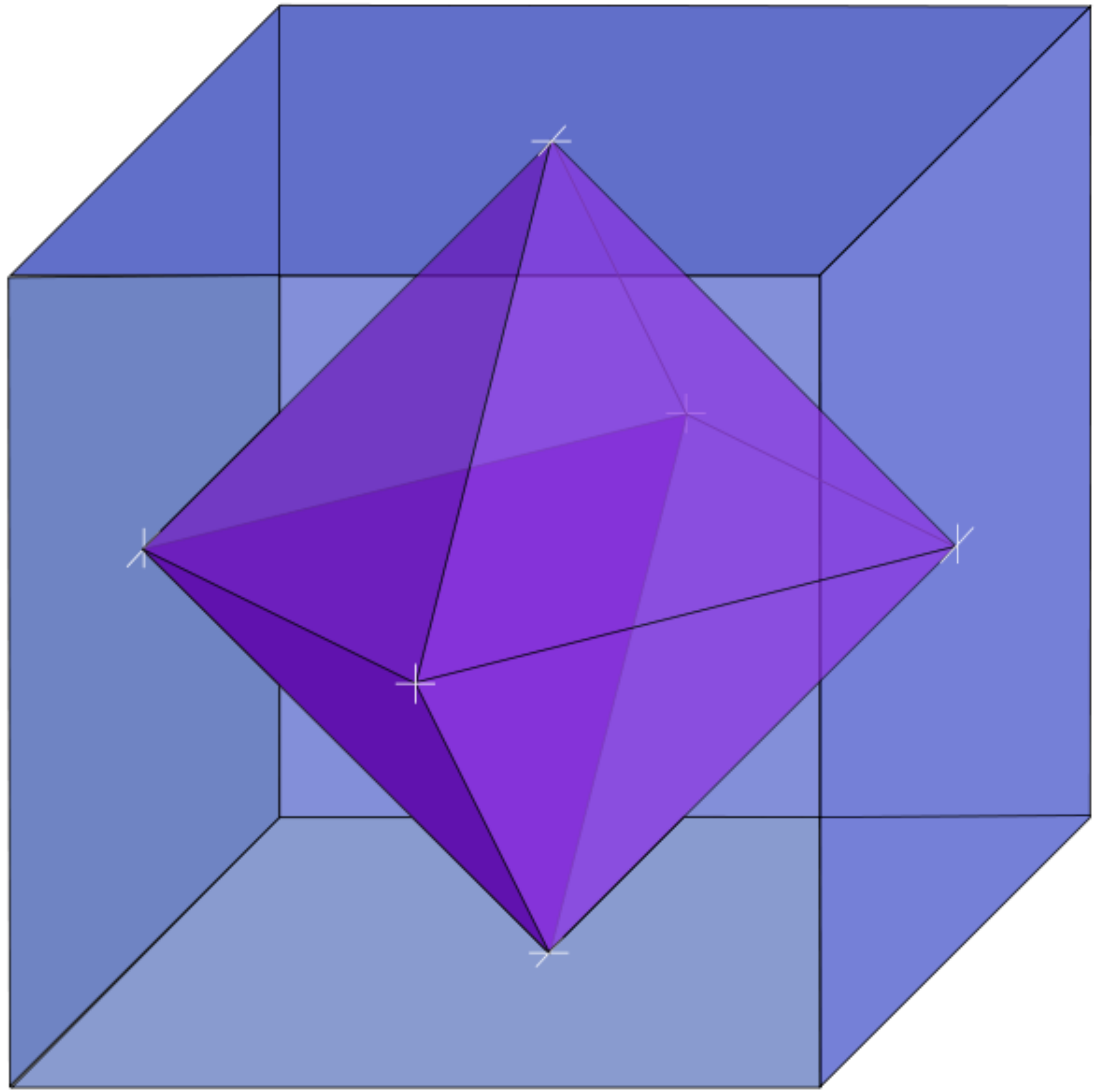
$$x^* = (x_1^*, x_2^*, \ldots\ldots, x_n^*)$$

then the dual also has an optimal solution,

$$y^* = (y_1^*, y_2^*, \ldots\ldots, y_m^*)$$

and

$$max \sum_j c_j\, x_j = min \sum_i b_i\, y_i$$



幾何学

Γεωμετρία

Géométrie

Geometry

Κανένας δεν εισάγει εκτός αν ξέρει τη γεωμετρία

Πλάτων

Nobody enters unless he knows Geometry

Plato

κανένας δεν παίρνει από εδώ εκτός αν ξέρει τη γεωμετρία

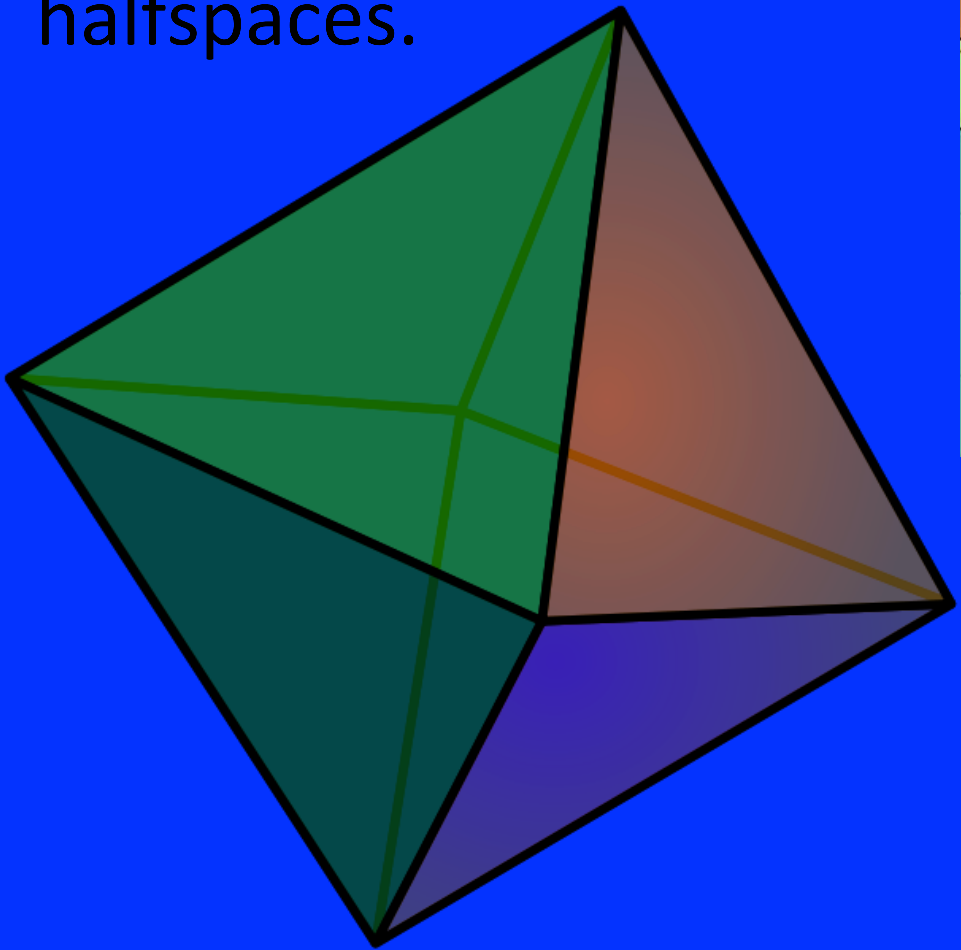
Jean-Louis L.

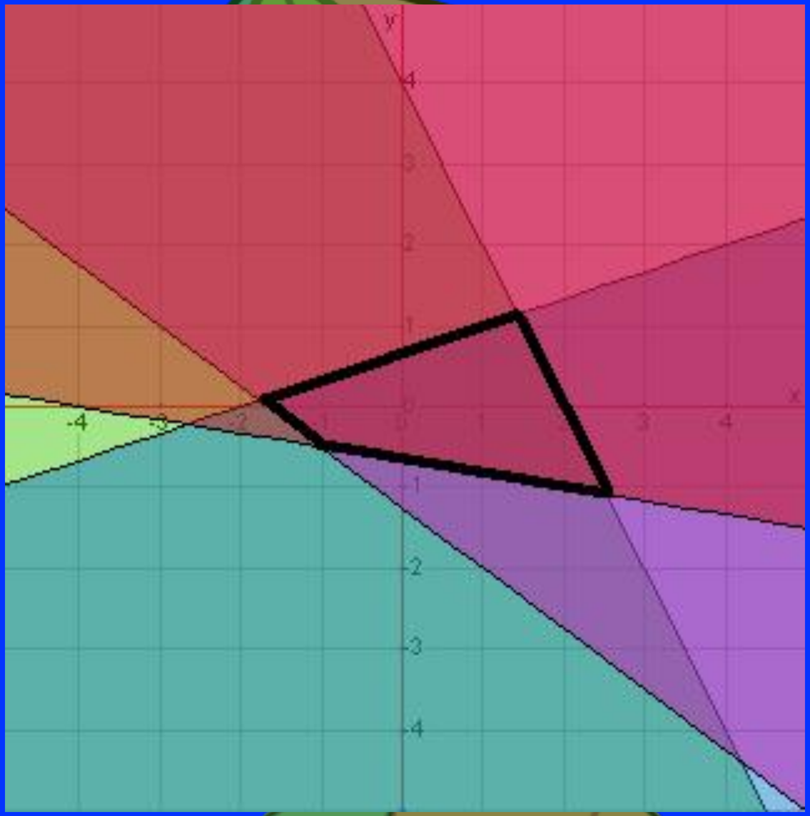
彼が幾何学を知っていなければだれも出ない

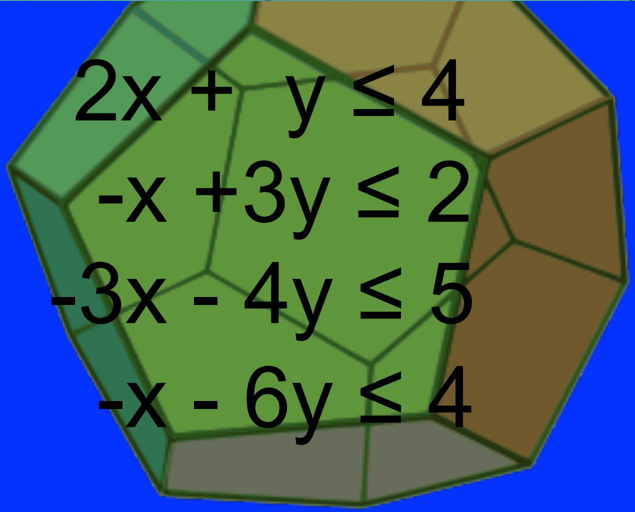
Personne ne sort s' il ne connait la Géométrie

Nobody gets out unless he knows Geometry

A polyhedron is defined as the intersection of a finite number of linear halfspaces.







$$
\begin{aligned}
2x + y &\le 4 \\
-x + 3y &\le 2 \\
-3x - 4y &\le 5 \\
-x - 6y &\le 4
\end{aligned}
$$

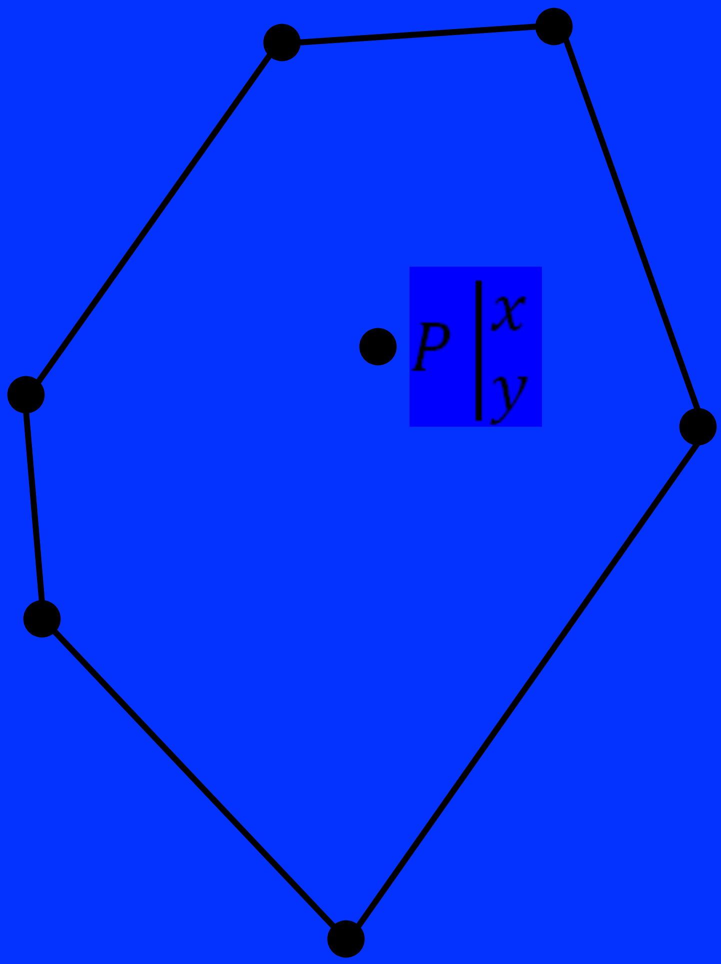
A polytope Q is defined as a convex hull of a finite collection of points.

$$x = \sum \lambda_i x_i$$

$$y = \sum \lambda_i y_i$$

$$\sum \lambda_i = 1$$

$$\lambda_i \geq 0$$



Minkowski(1896)-Steinitz(1916)-Farkas(1906)-Weyl(1935) Theorem

Q is a polytope if and only if it is a bounded polyhedron.

- Extension by Charnes & Cooper (1958)

"This classical result is an outstanding example of a fact which is completely obvious to geometric intuition, but wields important algebraic content and is not trivial to prove."

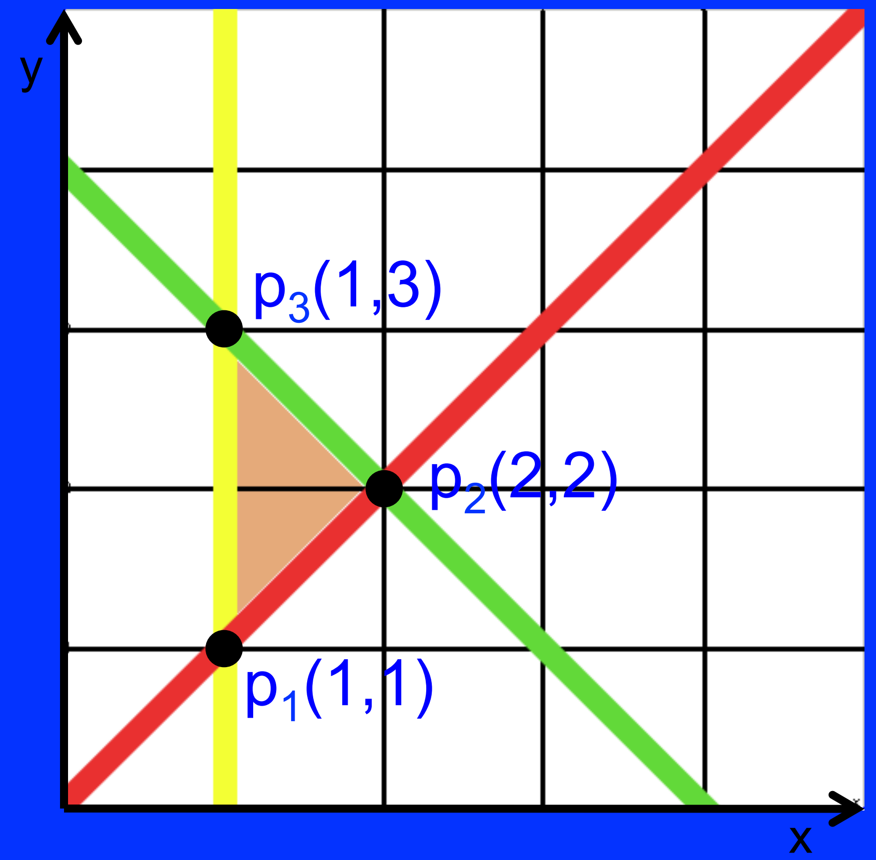
R.T. Rockafeller

A polytope Q is defined as a convex hull of a finite collection of points.

$x \leq 4 - y$

$x \geq 1$

$y \geq x$



A polyhedron is defined as the intersection of a finite number of linear halfspaces.

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